

Consistency of the mass variation formula for black holes accreting cosmological fluids

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We address the spherical accretion of generic fluids onto black holes. We show that, if the black hole metric satisfies certain conditions, in the presence of a test fluid it is possible to derive a fully relativistic prescription for the black hole mass variation. Although the resulting equation may seem obvious due to a form of it appearing as a step in the derivation of the Schwarzschild metric, this geometrical argument is necessary to fix the added degree of freedom one gets for allowing the mass to vary with time. This result has applications on cosmological accretion models and provides a derivation from first principles to serve as a base to the accretion equations already in use in the literature.

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I. INTRODUCTION

The accretion of matter onto black holes is one of the cornerstones of black hole astrophysics. In particular, spherical accretion in a cosmological context has also been considered [1] (see, for instance, the recent review by Carr [2] and references therein) and poses interesting problems for the survival of primordial black holes to the present epoch. Along with Hawking evaporation, it is essential on providing an accurate and thorough description for the evolution of the mass of both astrophysical and primordial black holes throughout the evolution of the universe.

The most common form of fluid which is considered on accretion models is a non-self-gravitating (test) perfect fluid, which can be safely used as a model for most types of ordinary and exotic matter, such as radiation, cold dark matter and dark energy. Several accretion models have been proposed for different types of fluids, and their connection with cosmological models is still an open problem.

The simplest accretion model is the classical Bondi model [3]. Being based on the non-relativistic continuity equation, it accurately describes accretion onto extense objects, such as stars. However, due to its Newtonian derivation, its accuracy is unclear when one considers small distances to a black hole event horizon, even within the simplest Schwarzschild solution. Another difficulty arises when using Bondi accretion of a fluid originated from a scalar field on a general relativistic scenario: it leads us to consider only the kinetic part of the field [4–6]. In the general case, the result is different when a fully relativistic approach is adopted for the accretion model.

This model can be refined if we consider the relativistic capture of particles by the black hole [7]. Essentially a gravitational scattering problem, it is most effective in describing the

accretion of radiation [8] by the use of a capture cross-section. However, this model, as well as the Bondi model, leads to problems when one considers a pressureless fluid [9].

A description of the fluid behavior outside the black hole horizon without the necessity of considering the individual particles of the field has been provided by Michel [10], based on the covariant conservation of the fluid’s energy-momentum tensor. It links the fluid evolution with the metric, allowing for a fully relativistic description of the fluid behavior. However, the Michel model says nothing about the actual energy transfer into the black hole, whose description remains left to the Bondi model.

A possible missing piece for the fully relativistic description of this accretion scenario has been provided in the work by Babichev *et al.* [11] and has since been used in the literature [12–17], through the use of a mass variation equation of the form

$$\frac{dm}{dt} = \mathcal{A} T_0^{-1}. \quad (1)$$

where \mathcal{A} is the area of the event horizon and T_0^{-1} is the (t, r) component of the mixed energy-momentum tensor of the accreted fluid. Although its seemingly obvious meaning, equation (1), as it is presented here, can be interpreted in two ways: **(a)** As a step towards the derivation of the Schwarzschild metric, coming directly from the Einstein equations for a spherically symmetric space-time [18]. In that specific case, it can be directly obtained from the fluid’s energy-momentum conservation. **(b)** As a first approximation to a fully relativistic continuity equation.

Both these interpretations suffer from conceptual problems. Assuming that equation (1) is obtained from the Einstein equations, then it would introduce an inconsistency with the conditions for the derivation of the metric itself. The Schwarzschild metric is static by construction, which would require the term \dot{m} to be identically zero. If we abandon this hypothesis, it would be necessary for consistency to also abandon the

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static metric and vacuum hypotheses and perform a full back-reaction analysis [19, 20]. This is why the so-called quasi-static approximation is generally adopted, and the mass evolution is assumed to be sufficiently slow for the metric to be considered static at each instant, with a slowly evolving black hole mass function.

Conversely, the continuity equation interpretation is also unsatisfactory, as it lacks a proper rigorous derivation from first principles and fundamental properties of the energy-momentum tensor.

In this work we provide a simple and exact geometric derivation of equation (1) and establish its range of validity, based on the properties of a space-time with a black hole under certain assumptions and a generic non-self-gravitating test fluid. The advantages of achieving such a formal derivation include not only a better understanding of its origins but it also provides one with the intuition from a simple case when treating other, more complicated scenarios.

Here we also adopt the quasi-static hypothesis, and we make no allusion to the field equations during the derivations. This decision is based on the fact that we understand that the black hole metric as derived from the *vacuum* field equations is now a background for the accretion process of a *test fluid*, and should take place from there without any further source terms. Another way of seeing that the Einstein equations should not be used during the derivation of quasi-static accretion is the fact that, once the metric is fixed to find the vacuum solution, we have no remaining constraints from the field equations to fix the degree of freedom in m . Therefore, the only self-consistent way to use the field equations to assess accretion would be through a full back-reaction analysis.

II. ENERGY FLOW THROUGH HYPER-SURFACES

Let $d^3\Sigma_v$ be an oriented hyper-surface volume element, parametrized by coordinates a, b and c [21]. This element may be cast in the space-time coordinates as

$$d^3\Sigma_v = \epsilon_{\nu\alpha\beta\gamma} \frac{\partial x^\alpha}{\partial a} \frac{\partial x^\beta}{\partial b} \frac{\partial x^\gamma}{\partial c} da db dc. \quad (2)$$

The four-momentum p^μ contained inside such a volume is given by the value of the energy-momentum which crosses the volume from past to future at a certain event. Then

$$p^\mu = \int_V T^{\mu\nu} d^3\Sigma_v. \quad (3)$$

If we take V to be the volume of the object and u^μ its four-velocity in its rest frame, then integrating equation (2) yields the oriented volume in this frame, $\Sigma_v = V u_v$, which allows us to write equation (3) as

$$p^\mu = V T^{\mu\nu} u_v. \quad (4)$$

The energy contained inside the volume V , measured in the object's rest frame, is the four-momentum projection onto the four-velocity

$$E = V T^{\mu\nu} u_\mu u_\nu. \quad (5)$$

If there is a flux of material through a limiting surface of the object during a proper time interval $\Delta\tau$, the four-momentum variation is given by the value of the energy-momentum which crosses the volume of another hyper-surface: the world-volume of the limiting surface during the interval $\Delta\tau$. We illustrate an example of the construction of this hyper-surface on a spherically symmetric spacetime with an event horizon located at $r_G = R$ on figure 1.

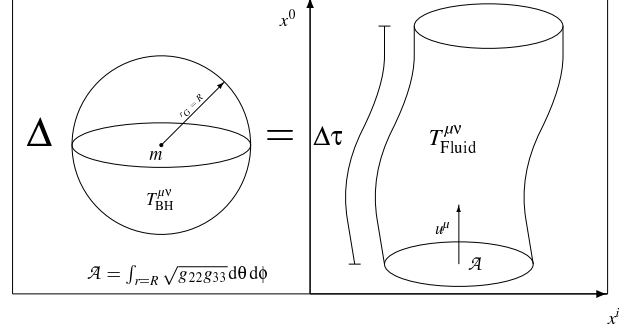


Figure 1. Four-momentum variation on the volume inside the event horizon of a spherically symmetric black hole by the flux of energy-momentum through the boundary of area \mathcal{A} . We refer to $T^{\mu\nu}_{\text{Fluid}}$ in the text simply as $T^{\mu\nu}$.

Thus, the oriented volume of the world sheet given by the surface of area \mathcal{A} during a proper time interval $\Delta\tau$ is given by equation (2)

$$\Sigma_v^{\text{BH}} = \mathcal{A} \Delta\tau \sigma_v \quad (6)$$

in which σ_v is a unit four-vector orthogonal to the world sheet. This orientation is a result of the product between the Levi-Civita symbol $\epsilon_{\nu\alpha\beta\gamma}$ and the one-forms tangent to the hyper-surface.

The variation of the object's four-momentum is then

$$\Delta p^\mu = \mathcal{A} \Delta\tau T^{\mu\nu} \sigma_v \quad (7)$$

and the energy variation measured in the object's rest frame is, as in (5),

$$\Delta E = \mathcal{A} \Delta\tau T^{\mu\nu} u_\mu \sigma_\nu. \quad (8)$$

III. ENERGY-MOMENTUM INSIDE THE EVENT HORIZON

The results up to this point are completely general. We now move on to the specific case when the energy-momentum contained in the interior region may be identified with the classical energy. This is true on asymptotically flat space-times.

Let us now assume that the exterior region is filled with a generic test fluid which flows along the radial direction towards the black hole interior through the event horizon. This allows us to define the volume (6) as the area of the event horizon \mathcal{A} times a proper time interval $d\tau$. When one moves

sufficiently far from the back hole, there can be associated with the system an energy, whose variation (8) is measured as

$$\frac{dE}{d\tau} = \mathcal{A} T^{\mu\nu} u_\mu \sigma_\nu. \quad (9)$$

A central point-like object of mass m in empty asymptotically flat space, as for example a Schwarzschild or Reissner-Nordström black hole, may be assigned an energy equal to the ADM mass [22], defined as the value of the Hamiltonian for a particular solution of the field equations at spatial infinity.

$$E = m \equiv \frac{1}{16\pi} \oint_S g^{\mu\nu} (g_{\mu\alpha,\nu} - g_{\mu\nu,\alpha}) dS^\alpha \quad (10)$$

where S is a 2-sphere with a radially oriented unit length area element dS^α . The integral must be taken in the limit where S approaches spatial infinity, and is only defined if the metric is asymptotically Euclidean [23].

By equating the energy variation inside the horizon to the energy variation due to the matter flux through the horizon surface, equation (9) becomes

$$\frac{dm}{d\tau} = \mathcal{A} T^{\mu\nu} u_\mu \sigma_\nu. \quad (11)$$

We assume now that the line element with which we are working is diagonal, as for example is the case of the static Schwarzschild metric in both isotropic and curvature coordinates, and of the non-static Vaidya [24] metric, as well as other solutions of physical interest [25–27]. We also assume that the central mass is a function only of time, to avoid arbitrariness when defining a metric with an extense central object [28]. Thus, we may cast the left-hand side of equation (11) in the rest frame as

$$\frac{dm}{d\tau} = \frac{dm}{dt} \frac{dt}{d\tau} = \frac{dm}{dt} \frac{1}{\sqrt{g_{00}}}. \quad (12)$$

Writing explicitly $\sigma_\mu = (0, 1, 0, 0)$ and using the four-velocity normalization in the rest frame $u^\mu u_\mu = 1$, whose result is

$$u^\mu = \left(\frac{1}{\sqrt{g_{00}}}, 0, 0, 0 \right); \quad u_\mu = (\sqrt{g_{00}}, 0, 0, 0). \quad (13)$$

we may then rewrite equation (11) as

$$\frac{dm}{dt} = g_{00} \mathcal{A} T^{01}. \quad (14)$$

By our diagonal metric assumption, we may finally express the mass variation as [11]

$$\frac{dm}{dt} = \mathcal{A} T_0^{01}. \quad (1)$$

IV. CONCLUSIONS

We have worked out in this paper a simple yet formal expression to deal with the problem of the accretion of a cosmological fluid onto a black hole. The formal derivation of

equation (1) completes the framework set up by Michel [10] in the test-fluid approximation of spherically symmetric accretion. The result justifies and clarifies some previous works and constitutes a reliable prescription which is fully consistent with the general relativistic conservation equations and does not introduce any conflicts with the Einstein equations.

In particular, it is now clear what are the requirements the metric must fulfill in order for equation (1) to be valid:

- There is a background solution to the field equations which constitutes a spherically symmetric black hole;
- The metric is asymptotically flat, so an ADM mass is well defined;
- The line element is diagonal;
- The accreted fluid is non self-gravitating;
- Accretion is quasi-static.

Equation (1) should not be interpreted as depending on local values of T_0^{01} , which is in general also a function of the radius, but must instead be evaluated in terms of global (asymptotic) features of the fluid and the black hole. However, by itself it does not provide us with enough constraints to determine those invariant values. To acquire the complete picture of the evolution, we must couple equation (1) with the conservation of the energy-momentum tensor and the four-momentum, as per the procedure derived by Michel [10], through the equations

$$T^{\mu\nu}_{;\nu} = 0 \quad (15)$$

$$u_\mu T^{\mu\nu}_{;\nu} = 0 \quad (16)$$

In some cases, a first integral of the latter can be obtained, eliminating the dependence with the radial coordinate [11]. In the particular case of the Schwarzschild metric accreting a perfect fluid, the expressions (15) and (16) have been fully worked out [10, 29], and the simplicity of the Schwarzschild line element does not require the derivation described here. In fact, the prescription for \dot{m} consisting of solving the system formed by equations (1), (15) and (16) is valid for more general non-static metrics and can be used for any situation which satisfies the above requirements. In principle, one might also relax the requirements of asymptotic flatness and the existence of an ADM mass, as is the case of the McVittie metric [30] and its generalizations [31], provided there can be defined an analogue to the black hole energy.

In particular, at least within this class of models, the question of whether a Schwarzschild black hole shrinks in mass when accreting a perfect phantom field has been clarified, and we now have presented an additional argument to this discussion, which has emerged in the literature due to some authors considering a classical energy transfer [2, 4, 32, 33]. More general cases, for example in which the phantom behavior results from viscosity terms and/or heat flows [31, 34–36], can in principle be handled safely in this framework.

More complicated fluids may lead to different results, even within this test fluid approximation. It has been pointed out

[15] that the mass increasing or decreasing due to the accretion process of a perfect fluid is directly related to the violation of the weak energy condition. However, this may not be true if non-ideal fluids are considered [19]. Another possibility is the modification of equation (1) due to quantum effects [37], which must be further explored within this framework.

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